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Abstract

We study the solutions of difference equations which approximate the single nonlinear conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1}$$

subject to

$$u(x,0) = u_o(x) , \quad u,f \in R^1 .$$

Approximations to (1) can be calculated by explicit difference schemes due to Lax and Wendroff [10,11,15]. We establish the existence of traveling waves of these difference schemes which are approximations to shock waves of (1). That is, there exist lattice functions for which the n^{th} iterate of the lattice function is equivalent to translation by a distance ($sn\Delta t$). The speed s is the same as the speed of the analogous shock solution of the differential equation.

1. Introduction

We are interested in the behavior of approximations to shocks which are solutions of

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1.1}$$

subject to

$$u(x,0) = u_o(x) , \qquad u,f \in R^1 .$$

First, we give a twenty-five cent tour of the theory of the partial differential equation.

The nonlinearity of (1.1) introduces phenomena which do not occur in the solutions of linear equations. In general, a solution $u(x,t)$ will develop jump discontinuities, which are called shocks, for initial conditions $u_o(x)$ which are infinitely differentiable. It is necessary to extend the class of solutions of (1.1) to admit these discontinuous solutions. An integrable function u is said to be a weak solution of (1.1) if it satisfies

$$\iint_{-\infty}^{\infty} [\psi_t u(x,t) + \psi_x f(u)]dxdt + \int_{-\infty}^{\infty} \psi(x,0)u_o(x)dx \tag{1.2}$$

for all functions ψ which are continuously differentiable and of compact support. Integration by parts shows that a solution $u(x,t)$ of (1.2) satisfies (1.1) in those regions where it has one continuous derivative with respect to each of x and t. Suppose that $u(x,t)$ has a jump discontinuity which is described by a smooth curve $x(t)$. Moreover, let $u_r(t)$ and $u_\ell(t)$ be the limits from left and the right of $u(x,t)$ at the discontinuity. By choosing a sequence ϕ_n of

nonnegative functions with supports which diminish to a point on the discontinuity, the definition of a weak solution implies the Rankine-Hugoniot relation

$$s(u_r(t) - u_\ell(t)) = f(u_r) - f(u_\ell) \qquad (1.3)$$

where $s = dx/dt$ is the speed of propagation of the shock.

In general, there is not a unique function which satisfies (1.2) for a given initial condition $u_o(x)$. The ambiguity results from spurious non-physical shocks being present. We give a condition which restricts the class of discontinuities which are admissible. We assume that $u_r < u_\ell$, then we require that

$$f(u) < (u_\ell - u)f(u_r) + (u - u_r)f(u_\ell) \qquad (1.4)$$

for u in the interval (u_r, u_ℓ). Geometrically, $f(u)$ lies below the chord in the $(u, f(u))$ plane connecting the points $(u_r, f(u_r))$ and $(u_\ell, f(u_\ell))$. If $u_r > u_\ell$ the relation is

$$f(u) > (u_r - u)f(u_\ell) + (u - u_\ell)f(u_r) \qquad (1.5)$$

for $u \in (u_\ell, u_r)$. Equations (1.4) and (1.5) are entropy conditions which we abbreviate to Condition E. They are due to Oleinik [23].

Hopf, Lax, and Oleinik [8,9,10,11,20] have studied the partial differential equation (1.1) in detail. Recently Quinn [22] has established that (1.1) is well posed in the $\mathcal{L}_1$ norm when Condition E is used to select the physically relevant shocks. Glimm has established the existence of weak solutions for all time when u and $f(\cdot)$ are vectors in R^n and $u_o(x)$ has small total variation. The

asymptotic properties of these solutions are studied in [5]. Nishida [17] has removed the condition that $u_o(x)$ be close to a constant for the equations of isothermal motion of a gas in Lagrangean coordinates. The twenty-five cent tour of the theory of the partial differential equation is complete. The reader with additional car fare is advised to switch to the recent survey article of Lax [24].

We study conservative difference schemes which are, those that can be written in the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\{g(u_{j+k}^n, u_{j+k-1}^n, \ldots, u_{j-k+1}^n) - g(u_{j+k-1}^n, \ldots, u_{j-k}^n)\} \tag{1.6}$$

where $u_j^n = u(n\Delta t, j\Delta x)$ and $g(\cdot,\cdot,\ldots,\cdot)$ is a function of 2k variables. The equation (1.6) will be a consistent approximation to (1.1) if $g(u,u,\ldots,u) = f(u)$. A prototype of an equation of the form (1.1) which we previously discussed is

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 . \tag{1.7}$$

Lax [10] proposed the scheme

$$u_j^{n+1} = \frac{u_{j+1}^n}{2} + \frac{u_{j-1}^n}{2} - \frac{\Delta t}{2\Delta x}\{f(u_{j+1}^n) - f(u_{j-1}^n)\} \tag{1.8}$$

to approximate (1.7). This equation can be written in the form (1.6) taking

$$g(a,b) = \frac{b-a}{2} \cdot \frac{\Delta x}{\Delta t} + \frac{a^2+b^2}{2} . \tag{1.9}$$

The right-hand side of (1.8) is a monotone increasing function of each of u_{j+1}^n and u_{j-1}^n so long as $|\frac{\Delta t}{\Delta x} u_j^n| < 1$ for all j. The bound on $\Delta t/\Delta x$ is the Courant-Friedrichs-Lewy stability condition for the linearized difference scheme.

In general, we require that the right-hand side of (1.6) be a monotone increasing function of each of the variables $u_{j+k}^n, \ldots, u_{j-k}^n$.[1] We give the right-hand side of (1.6) a name rewriting the equation as

$$u_j^{n+1} = G(u_{j+k}^n, u_{j+k-1}^n, \ldots, u_{j-k}^n) \ . \tag{1.10}$$

Note that $G(u, u, \ldots, u) = u$.

We construct traveling wave solutions of (1.6) with limits u_r and u_ℓ at plus and minus infinity respectively. We require that $f(\cdot)$ satisfy Condition E across the interval (u_r, u_ℓ). In the following, we assume $u_r < u_\ell$. The case $u_r > u_\ell$ can be reduced to this case by symmetry. A traveling wave u_x which moves with speed s satisfies the difference equation

$$u_{x-s(\Delta t/\Delta x)} = G(u_{x+k}, u_{x+k-1}, \ldots, u_{x-k}) \ . \tag{1.11}$$

The speed s is related to u_r and u_ℓ by the Rankine-Hugoniot relation.

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[1] That (1.9) does not depend explicitly on u_j^n is a technical problem that can be removed by iterating the equation once. The resulting relation is of the form (1.6) and defines u_j^{n+2} as a function of u_{j-2}^n, u_j^n, and u_{j+2}^n. Leibniz's rule for calculating the partial derivatives of a composite function implies that the iterated relation is a monotone increasing function of each of u_{j-2}^n, u_j^n, and u_{j+2}^n.

Condition E is the weakest condition that one can expect to imply the existence of traveling waves for all values of $\Delta t/\Delta x$. Consider the closely related problem of constructing traveling waves of

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \varepsilon \frac{\partial^2}{\partial x^2} u$$

when $\varepsilon > 0$. Traveling waves exist and are continuous, Foy [4]. To obtain the relation they satisfy write $u(x,t) = v(x-st)$ and substitute to obtain

$$-sv' + [f(v)]' = \varepsilon v'' .$$

Integrate this relation once,

$$v' = \frac{sv - f(v)}{\varepsilon} + \text{constant} . \tag{1.12}$$

The boundary conditions $u_{+\infty} = u_r$ and $u_{-\infty} = u_\ell$ determine the constant as that value at which u_r and u_ℓ are critical points of (1.12), i.e., the right side vanishes when v equals u_r or u_ℓ. The Rankine-Hugoniot condition implies the right side vanishes at u_r if and only if it vanishes at u_ℓ. Note that no orbit running from u_r toward u_ℓ can get past the first zero of the right side of (1.11), that is, past a point where Condition E is violated.

The traveling waves we construct are fixed points of an operator T defined by

$$(Tv)_x = G(v_{x+k-\eta}, v_{x+k-1-\eta}, \ldots, v_{x-k-\eta})$$

where $\eta = -s\Delta t/\Delta x$. In a subsequent paper, we shall show that these fixed points are stable in the following strong sense. If u_o is a function defined on the integers which takes on values for which $G(\cdot,\cdot,\ldots,\cdot)$ is a monotone increasing function of each of its arguments, then $T^n u_o$ converges in L_1 to a traveling wave.

2. Existence of Traveling Waves When η is Rational

We fix a choice of u_r and u_ℓ across which $f(\cdot)$ satisfies Condition E. The speed s is determined from the Rankine-Hugoniot relation on u_r, u_ℓ, and $f(\cdot)$. The parameter η is then a function of $\theta = \Delta t/\Delta x$. We construct in this paper a traveling wave for each choice of θ in the interval $(0,\theta_{max})$ where θ_{max} is the largest value of θ for which each of the partial derivatives $\frac{\partial G}{\partial u_j}(u_k,u_{k-1},\ldots,u_{-k})$, $k \geq j \geq -k$, is nonnegative when all u_j, $k \geq j \geq -k$ lie in the interval $[u_r,u_\ell]$.

The construction of traveling waves is done in two parts. First we construct waves when θ is chosen so that $\eta = s\theta$ is rational. Traveling waves which satisfy (1.12) when $s\theta$ is irrational will be constructed as the limits of traveling waves which satisfy (1.12) with values of $s\theta_i$ rational which converge to $s\theta$.

In the remainder of this section, η is rational. We seek solutions of

$$u_{x+\eta} = G(u_{x+k},u_{x+k-1},\ldots,u_{x-k}) \tag{2.1}$$

such that $u_{+\infty} = u_r$ and $u_{-\infty} = u_\ell$. We iterate (2.1) once and obtain

the equation

$$u_{x+2\eta} = G^{(1)}(u_{x+k'}, u_{x+k'-1}, \ldots, u_{x-k'}) \ .$$

$G^{(1)}$ is a monotone increasing function of each of its arguments as G is. We may iterate (2.1) a finite number of times so that N_η is an integer, maintaining the form of the equation after each iteration. Hence, without loss we consider (2.1) when η is an integer. The Courant-Friedrichs-Lewy condition requires that $|\eta| < k$. We fix u_r, u_ℓ, s and $f(\cdot)$ and consider functions u_x defined on the integers. We seek traveling waves as solutions of (2.1) with limits at plus and minus infinity, $u_{+\infty} = u_r$ and $u_{-\infty} = u_\ell$.

We prove the following

Theorem 1: (η rational) Let $f(\cdot)$ satisfy Condition E across (u_r, u_ℓ) where u_r and u_ℓ determine a shock solution of (1.1) which travels with speed s. Suppose that $\theta = \Delta t/\Delta x$ is chosen so that $s\Delta t/\Delta x$ is rational and is smaller in magnitude than k. Moreover, we suppose that the right-hand side of (2.2) is a monotone increasing function of each of $u_{x+k}, u_{x+k-1}, \ldots, u_{x-k}$ when its arguments take on value in the interval $[u_r, u_\ell]$. Then for each number $u_o \in (u_r, u_\ell)$, there is a unique function defined on the integers which takes on the value u_o at $x = 0$ and which satisfies

$$u_{x+\eta} = u_x - \frac{\Delta t}{\Delta x}\{g(u_{x+k}, u_{x+k-1}, \ldots, u_{x-k+1}) - g(u_{x+k-1}, u_{x+k-2}, \ldots, u_{x-k})\} \ . \tag{2.2}$$

The solution u_x is a monotone function of x and depends continuously at each value of x on u_o.

The monotonicity of the right side of (2.2), which we have named $G(\cdot,\cdot,\ldots,\cdot)$ previously, is central to the proofs which follow. We collect the consequences of this assumption that we will need later. We name the variables on which G depends writing it as $G(u_k,u_{k-1},\ldots,u_{-k})$. The partial derivatives with respect to each of $u_k,u_{k-1},\ldots,u_{-k}$ are positive. Hence, writing $g = g(w_k,w_{k-1},\ldots,w_{-k+1})$ and $\frac{\partial g}{\partial w_k} = g_k$, it follows that

$$\delta_j - \theta(g_j - g_{j+1}) > 0 \quad \text{for} \quad k \geq j \geq -k$$

where $\delta_j = 1$ if $j = 0$, and 0 otherwise. Moreover, $g_{k+1} = g_{-k} = 0$. From $g_{k+1} = 0$, it follows that

$$-\theta g_k > 0$$

and by induction, $-\theta g_j > -\theta g_{j+1}$, $j > 0$ implies

$$-\theta g_j > 0 \quad \text{for} \quad k \geq j > 0 \;. \tag{g1}$$

At $j = 0$, we obtain $(1 - \theta g_o) > -\theta g_1 > 0$. moreover, by induction it follows that

$$(1 - \theta g_j) > 0 \quad \text{for} \quad -k \leq j \leq 0 \;. \tag{g2}$$

Additional inequalities can be obtained by considering first that $\theta g_{-k} > 0$ and then proceeding by induction for $j > -k$. We need only the relations (g1) and (g2).

We give now an estimate for the difference $G(u_{x+k},u_{x+k-1},\ldots,u_{x-k}) - G(v_{x+k},v_{x+k-1},\ldots,v_{x-k})$. We use the mean value theorem and write the difference as

$$\int_0^1 \sum_{j=-k}^{k} \frac{\partial G}{\partial u_j} (\beta\, u_x^* + (1-\beta)v_x^*)(u_{x+j} - v_{x+j})d\beta$$

where $u_x^* = (u_{x+k}, u_{x+k-1}, \ldots, u_{x-k})$. The partial derivatives are positive. Hence the absolute value of the difference is the above expression but with $(u_{x+j} - v_{x+j})$ replaced by $|u_{x+j} - v_{x+j}|$. Using the mean value theorem again, we get

$$\begin{aligned} &\left|G(u_{x+k}, u_{x+k-1}, \ldots, u_{x-k}) - G(v_{x+k}, v_{x+k-1}, \ldots, v_{x-k})\right| \\ &\leq |u_x - v_x| - \theta\nabla g(\gamma_x)\cdot|\hat{u}_x - \hat{v}_x| + \theta\nabla g(\gamma_{x-1})\cdot|\hat{u}_{x-1} - \hat{v}_{x-1}| \end{aligned} \tag{P}$$

where γ_x is a point on the chord joining $\hat{u}_x = (u_{x+k}, u_{x+k-1}, \ldots, u_{x-k-1})$ and $(v_{x+k}, v_{x+k-1}, \ldots, v_{x-k+1})$. Inequality (P) is an equality if and only if each of $u_{x+k} - v_{x+k}$, $u_{x+k-1} - v_{x+k-1}, \ldots$ and $u_{x-k} - v_{x-k}$ has the same sign.

Prior to proving that a traveling wave exists, we prove that, assuming existence, the solution must be a monotone function of x and that a solution depends uniquely on the value which it assumes at a single point. Subsequent to these proofs we construct traveling waves.

Lemma 2.1. Let u_x and v_x be two traveling waves, such that $u_x, v_x \in [u_r, u_\ell]$ for all x, then u_x and v_x agree at one point of the lattice if and only if they agree at every point. If $u_x > v_x$ at a single point, then $u_x > v_x$ for all x.

Proof: By assumption

$$u_{x+\eta} = G(u_{x+k}, u_{x+k-1}, \ldots, u_{x-k})$$

and

$$v_{x+\eta} = G(v_{x+k}, v_{x+k-1}, \ldots, v_{x-k})$$

then by inequality (P)

$$|u_{x+\eta} - v_{x+\eta}| \leq |v_x - u_x| - \theta\nabla g(\gamma_x)\cdot|\hat{u}_x - \hat{v}_x| + \theta\nabla g(\gamma_{x-1})\cdot|\hat{u}_{x-1} - \hat{v}_{x-1}| \ .$$

Sum this inequality from x = -M to +M, and eliminate the common terms,

$$\sum_{x>M}^{M+\eta} |u_x - v_x| \leq \sum_{x=-M}^{x<-M+\eta} |v_x - u_x| - \theta\nabla g(\gamma_M)\cdot|\hat{u}_M - \hat{v}_M| + \theta\nabla g(\gamma_{-M})\cdot|\hat{u}_{-M} - \hat{v}_{-M}| \ .$$

As $M \to +\infty$, each of the terms goes to zero. We obtain

$$0 \leq 0 \ .$$

This implies that Inequality (P) was an equality for each value of x. Hence $v_{x+k} - u_{x+k}$, $u_{x+k-1} - v_{x+k-1}, \ldots$ and $v_{x-k} - u_{x-k}$ are of the same sign for each value of x. The sign may depend on x. The difference $v_x - u_x$ cannot go from positive to negative without being zero. If $u_x = v_x$ for a single value of x, then

$$0 = u_x - v_x = \int_0^1 \frac{\partial G}{\partial j}\,(\theta u^*_{x-\eta} + (1-\theta)v^*_{x-\eta})\cdot(u^*_x - v^*_x)d\theta$$

but, as $\partial G/\partial j > 0$, it must be that $u^*_t = v^*_t$ for $t \in [x-\eta-k, x-\eta+k]$. Proceeding by induction, it follows that $u_x \equiv v_x$ if $u_x = v_x$ at a single point. Thus if $u_x > v_x$ at a single point, $u_x > v_x$ for all x. This completes the proof.

Consider that u_x is a traveling wave, then u_{x+1} is also a traveling wave. Hence, $u_x - u_{x+1}$ is of the same sign for all x. Therefore we have

Lemma 2.2. Let u_x be a traveling wave such that $u_x \in [u_r, u_\ell]$, then u_x is a monotone function of x and lies in the interval (u_r, u_ℓ).

We now proceed to the construction of traveling waves. We study the fixed points of the operator T_r defined on functions whose domain is the integers greater than $-(k+\eta)$. For $x+\eta > 0$ we define

$$(T_r v)_{x+\eta} = G(v_{x+k}, v_{x+k-1}, \ldots, v_{x-k}) \tag{2.3}$$

T_r is the identity on the integers x such that $-k-\eta < x \leq 0$. We define a fixed point of T_r to be a function u_x defined on I^+, the integers greater than $-k-\eta$, which satisfies

$$v_x = (T_r v)_x$$

and has the limit $u_{+\infty} = u_r$.

A one-parameter family of fixed points of T_r can be extended to a traveling wave. We identify this family as those fixed points of T_r which can be "patched" to a fixed point of T_ℓ. T_ℓ is analogous to T_r and is defined on functions with domain, I^-, the integers less than $k-\eta$ by

$$(T_\ell w)_{x+\eta} = G(w_{x+k}, w_{x+k-1}, \ldots, w_{x-k})$$

when $x+\eta < 0$ and $(T_\ell w)_x = w_x$ when $0 \leq x < k-\eta$. We require a fixed point of T_ℓ to have the limit $w_{-\infty} = u_\ell$. We define the initial value of a function with domain equal to I^+ to be the values which it assumes on the interval $-k-\eta < x \leq 0$. Here and in the following, x assumes values in the integers. The objective of the following lemmas is to establish

Lemma 2.3. Let ξ_x be an initial value on I^+ such that $u_r \leq \xi_x \leq u_\ell$ for $-k-\eta < x < 0$ and $u_r \leq \xi_x < u_\ell$, for $x = 0$, then there exists a unique fixed point u_x of T_r with ξ_x as initial value. For a fixed value of x, u_x is a continuous function of ξ. Moreover, $u_r \leq u_x < \max \xi_x$ for $x > 0$.

An analogous result is true for T_ℓ. The initial value of a function defined in I^-, is its value on $0 \leq x < k-\eta$. We state

Lemma 2.4. Let ζ_x be an initial value on I^- such that $u_r \leq \zeta_x \leq u_\ell$ for $0 < x < k-\eta$ with $u_r < \zeta_x \leq u_\ell$, at $x = 0$, then there exists a unique fixed point w_x of T_ℓ with ζ_x as initial value. For a fixed value of x, w_x is a continuous function of ζ. Moreover $\min \zeta_x < w_x \leq u_\ell$.

With these two lemmas we can construct traveling waves when η is rational. We prove Theorem 1 now assuming Lemmas 2.3 and 2.4.

Fix $u_0 \in (u_r, u_\ell)$ and let ξ_x satisfy $u_r \leq \xi_x \leq u_\ell$, $-k-\eta < x < 0$, with ξ_x at $x = 0$ equal to u_0. Then there exists a fixed point of T_r, $u_x(\xi)$ such that $u_r \leq u_x < u_\ell$ for all x. The function u_x restricted to the interval $[0, k-\eta)$ is an initial value of a fixed point w_x of T_ℓ. Moreover, $u_r < w_x \leq u_\ell$ for all x and for $-k-\eta < x < 0$ in particular. The fixed point w_x determined in this manner is a continuous function of ξ_x. Moreover, for all ξ, $u_r \leq \xi_x \leq u_\ell$ for $-k-\eta < x < 0$, it follows that $u_r \leq w_x(\xi) \leq u_\ell$ for $-k-\eta < x < 0$. Hence the Brouwer fixed point theorem implies there is a fixed point ξ_x^*. Define a function v_x on I^+ as $u_x(\xi^*)$ and on I^- as $w_x(\xi^*)$. As $w_x(\xi^*) = \xi^*$, the definition is the same on $I^+ \cap I^-$. Notice that v_x is a traveling wave of the difference scheme and takes on the value u_0 at $x = 0$, which was chosen arbitrarily. We gather this into

Lemma 2.5. Theorem 1 is a consequence of Lemmas 2.3 and 2.4.

Proof: All that remains is to show that a traveling wave depends continuously on the value it assumes at any point.

Let v_x^i be the traveling wave which takes on the value u_i at $x = 0$ and suppose that $u_i \to u_o$. We may suppose that $u_+ < u_i < u_-$ for all i and that u^+ and $u^- \in (u_r, u_\ell)$. Then by Lemma 2.1,

$$u_x^- < v_x^i < u_x^+$$

where u_x^- and u_x^+ are traveling waves which u^- and u^+ determine respectively. Extract a subsequence, if necessary, so that $v_x^i \to v_x$. Then

$$v_{x+\eta} = G(v_{x+k}, v_{x+k-1}, \ldots, v_{x-k})$$

and moreover, $u_r = u^-_{+\infty} \leq v_{+\infty} \leq u^+_{+\infty} = u_r$. Similarly, $v_{-\infty} = u_\ell$. This finishes the proof.

We now turn to the proof of these Lemmas. We assume that all functions defined on I^+ take on their values in the interval $[u_r, u_\ell]$.

First, we show that a fixed point of T_r depends uniquely on its initial value. This we do directly from inequalities (g1) and (g2). To establish the existence of fixed points, we consider the set $\mathcal{A}$ of $a \in [u_r, u_\ell]$ such that every initial value for which

$$u_r \leq \xi_x \leq a \quad \text{when} \quad x < 0$$

and

$$u_r \leq \xi_x < a \quad \text{for} \quad x = 0$$

determines a fixed point of T_r. We show that $\mathcal{a}$ is both open and closed in the interval $[u_r, u_\ell]$. Using a continuity argument, Lemma 2.3 follows from the existence of fixed points with initial values close to u_r. Lemma 2.4 then follows by symmetry.

Lemma 2.6. (Unique Dependence on Initial Values) Let v_x and w_x be fixed points of T_r, which have the same initial value, then v_x is equal to w_x for all x.

Proof: Difference the relations that v_x and w_x satisfy, apply inequality (P), and sum from $x+\eta > 0$ to $x+\eta = M$. Eliminate the common terms from the sums on each side of the inequality and let $M \to +\infty$. The result is

$$\sum_{x>0}^{x\leq k-\eta} |v_x - w_x| - \theta\nabla g(\gamma_o)\cdot|\hat{v}_o - \hat{w}_o| \leq 0$$

where γ_o is a point in R^{2k} on the chord between $\hat{v}_o$ and $\hat{w}_o$. Using the assumption that $v_x = w_x$ for $x \leq 0$, this becomes

$$\sum_{x>0}^{x\leq k-\eta} \alpha_x\cdot|v_x - w_x| \leq 0$$

where

$$\alpha_x = \begin{cases} -\theta \dfrac{\partial g}{\partial u_{x+\eta+1}}(\gamma_o) & \text{for} \quad k-\eta > x > -\eta \\[2ex] 1-\theta \dfrac{\partial g}{\partial u_{x+\eta+1}}(\gamma_o) & \text{for} \quad 0 < x \leq -\eta \end{cases}$$

That $\delta_x > 0$ follows from (g1) and (g2) and hence $v_x = w_x$ for

$0 < x \leq k-\eta$. Translation to the left on I^+ maps fixed points into fixed points. Thus $v_x = w_x$ for $x > 0$ by induction.

We now give an estimate on the values of a fixed point in terms of the initial value. This is a maximum principle and does not require that the lattice function v_x has a limit at $x = +\infty$.

<u>Lemma 2.7</u>. (Maximum Principle) Let v_x satisfy

$$v_{x+\eta} = G(v_{x+k}, v_{x+k-1}, \ldots, v_{x-k}) \quad \text{for} \quad x+\eta > 0$$

$v_x \in (u_r, u_\ell)$ for all x, then v_x assumes its maximum (minimum) at a value $x_o > 0$, if and only if v_x is a constant. No condition is imposed on the behavior of v_x as $x \to +\infty$.

<u>Proof</u>: For $x_o > 0$,

$$v_{x_o} = G(v_{x_o+k-\eta}, v_{x_o+k-\eta-1}, \ldots, v_{x_o-k-\eta}) \,. \tag{2.4}$$

Moreover, assume without loss that $v_{x_o} \geq v_x$ for all x. G is a strictly monotone increasing function of each of the arguments. Hence

$$v_{x_o} \leq G(v_{x_o}, v_{x_o}, \ldots, v_{x_o}) = v_{x_o}$$

and the inequality is strict unless $v_{x_o-\eta+j} = v_{x_o}$ for $-k \leq j \leq k$. But (2.4) is a (nonlinear) recursion. If a solution is constant for any 2k consecutive values, the solution is constant for all x. This completes the proof.

We prove a weak result on the existence of fixed points. The result implies that the same inequality on each component on two initial values persists to the traveling waves.

Lemma 2.8. (Partial Ordering of Fixed Points) Let $u_r = \xi_x$ for $-k-\eta < x \leq 0$ be the initial value of a fixed point u_x. For each initial value ζ_x, $u_r \leq \zeta_x \leq \xi_x$, there is a fixed point w_x of T_r which takes on the initial value ζ_x and $w_x \leq v_x$ for all x.

Proof: By the maximum principle, $u_x > u_r$ for $x > 0$, unless $\xi_x \equiv u_r$. Therefore, the lattice function

$$w_x^o = \begin{cases} \zeta_x & \text{if} \quad x \leq 0 \\ u_r & \text{otherwise} \end{cases}$$

is bounded from above by u_x. Define w_x^n to be the n^{th} iterate of w_x^o under T_r. As G is a monotone increasing function of each of its arguments

$$w_{x+\eta}^1 = G(w_{x+k}^o, w_{x+k-1}^o, \ldots, w_{x-k}^o) \geq u_r$$

for $x+\eta > 0$. Moreover, by induction

$$u_r \leq w_x^n \leq w_x^{n+1} \leq u_x \ .$$

Thus the limit w_x^n as $n \to \infty$ exists which we write as w_x and satisfies

$$u_r \leq w_x \leq u_x \ .$$

Hence $w_{+\infty} = u_r$, and the proof is complete.

The continuous dependence of a fixed point upon its initial value now follows.

Lemma 2.9. (Continuous Dependence on Initial Values) Let ξ_x^i be a sequence of initial values such that

$$u_r \leq \xi_x^i \leq \zeta_x$$

where ζ_x is an initial value that determines a fixed point of T_r, then if $\xi_x^i \to \xi_x$ for each value of x, it follows that $v_x(\xi^i) \to v_x(\xi)$ pointwise, where $v_x(\xi^i)$ is the fixed point of T_r that ξ^i determines.

Proof: Because of the monotonicity of fixed points of T_r, it follows that

$$u_r \leq v_x(\xi^i) \leq v_x(\zeta) .$$

Passing to a subsequence if necessary, it follows that $v_x(\xi^i) \to w_x$. But

$$u_r \leq w_x \leq v_x(\zeta)$$

from which it follows that $w_{+\infty} = u_r$. The uniqueness result implies $w_x = v_x(\xi)$.

We define $\mathcal{E}(a)$ to be a set of initial values, ξ_x, such that

$$u_r \leq \xi_x \leq a \quad \text{for} \quad -k-\eta < x < 0$$

and

$$u_r \leq \xi_x < a \quad \text{when} \quad x = 0 \ .$$

We are interested in the supremum of the set $\mathcal{A}$ of $a \in [u_r, u_\ell]$ such that every initial value in $\mathcal{E}(a)$ admits a fixed point of T_r. Call this point a^*. Then $\mathcal{A} = [u_r, a^*)$ or $[u_r, a^*]$ by Lemma 2.9. We first show that $\mathcal{A}$ is closed.

Lemma 2.10. ($\mathcal{A}$ is closed) Let a^* and $\mathcal{A}$ be as above then every initial value in $\mathcal{E}(a^*)$ determines a fixed point of T_r.

Proof: Let $a_i \uparrow a^*$ where $a_i \in \mathcal{Q}$. Suppose ξ_x is an initial value in $\mathcal{E}(a^*)$. Hence $u_r \leq \xi_o < a^*$ and we may suppose that $\xi_x = a^*$ for some $x \in (-k-\eta, 0)$. We proceed as before setting

$$u_x^o = \begin{cases} \xi_x & \text{if} \quad x \leq 0 \\ u_r & \text{if} \quad x > 0 \end{cases} .$$

Then $u_x^n = T_r^n u_x^o \uparrow u_x$, which satisfies

$$u_{x+\eta} = G(u_{x+k}, u_{x+k-1}, \ldots, u_{x-k}) .$$

Moreover as $\xi_o < a^*$ and $\xi_x = a^*$ for some x the maximum principle implies that $u_x < a^*$ for all $x > 0$ and thus u_x restricted to $(0, k+\eta]$ is an initial value ζ which lies in $\mathcal{E}(a^i)$ for i sufficiently large. Hence, there exists a fixed point of T_r, $\hat{w}_x$, with initial value ζ. Set $w_x = \hat{w}_{x-(k+\eta)}$. Then $w_x > u_x^o$ for $x > 0$ and as w_x bounds u_x^n on the interval $(0, k+\eta]$ it follows that $u_x^n \leq w_x$ for $x > 0$. Hence $u_r \leq u_x \leq w_x$ and $u_{+\infty} = u_r$. This completes the proof.

We now turn to characterizing a^*.

Lemma 2.11. Let $a > u_r$. If, there exists a fixed point of T_r with initial value $\xi_x \equiv a$, then $\mathcal{Q}$ contains an interval about a.

Proof: We assume that there is a fixed point u_x of T_r with initial value $\xi_x \equiv a$. As u_x is not a constant, it follows from the maximum principle that $u_x < a$ for $x > 0$. The restriction of u_x to $(0, k+\eta]$ is an initial value ζ which lies in $\mathcal{E}(a-\varepsilon)$ for some ε positive and sufficiently small. The dimension of $\mathcal{E}(a-\varepsilon)$ is $k+\eta$.

Consider the equation

$$v_{x+\eta} = G(v_{x+k}, v_{x+k-1}, \dots, v_{x-k})$$

as a nonlinear recursion of order 2k. The process of calculating the value of v_{x-k} as a function of $v_{x+k}, v_{x+k-1}, \dots, v_{x-k+1}$ is a function $\mathcal{L}$ in the usual way from $R^{2k} \to R^{2k}$, which is well defined locally so long as $\frac{\partial G}{\partial u_k}(\hat{v}) \neq 0$. This makes $\mathcal{L}$ an open mapping. The continuous dependence of a fixed point on its initial value implies that the set of vectors in R^{2k} near $(v_1, v_2, \dots, v_{2k})$ which generates fixed points of T_r is a manifold of dimension $k+\eta$. Call this set $\mathcal{M}$. Then $\mathcal{L}^{k+\eta}(\mathcal{M})$ is of dimension $k+\eta$, because $\mathcal{L}$ is an open mapping. Fixed points of T_r depend uniquely on their first $k+\eta$ components. The projection of $\mathcal{L}^{k+\eta}(\mathcal{M})$ onto its first $k+\eta$ components must be of dimension $k+\eta$. This completes the proof.

If we suppose that a^* is greater than u_r, this lemma enables us to determine a^*. Select a sequence $a_i \uparrow a^*$ then the fixed points u_x^i such that $u_x^i \equiv a^i$ for $-k-\eta < x \leq 0$ satisfy

$$u_r \leq u_x^i \leq u_x^{i+1} \leq a^*$$

for all x and i. Hence $u_x^i \uparrow w_x$ at each value of x. If w_x is not a constant then $w_x < a^*$ for $x > 0$ and as before w_x is a fixed point of T_r.

Each of u_x^i is a fixed point of T_r. Sum the relation

$$u_{x+\eta}^i = G(u_{x+k}^i, u_{x+k-1}^i, \dots, u_{x-k}^i)$$

from $x > 0$ to $x = +M$, and remove the terms in common on both sides of the relation. Let $M \to +\infty$. It then follows that

$$\sum_{x>0}^{<k-\eta} u_x^i = \sum_{x>-\eta}^{<k-\eta} u_x^i - \theta g(u_{k-\eta-1}^i, u_{k-\eta-1}^i, \ldots, u_{-k-\eta}) \tag{2.5}$$

$$- \theta g(u_r, u_r, \ldots, u_r) + \eta u_r \ .$$

Note $g(u,u,\ldots,u) = f(u)$ and let $u_x^i \to w_x$. As $w_x = a^*$, it follows that

$$\eta(a^* - u_r) = \theta[f(u_r) - f(a^*)] \ .$$

Recall $\eta = -s\theta$ and $u_r \leq a^* \leq u_\ell$. Then a^* is the solution of

$$s(a^* - u_r) = f(a^*) - f(v_r) \ .$$

Condition E implies that u_ℓ is the first solution of this equation greater than u_r.

By the continuity method we have proved

Lemma 2.12. If there exists an $\varepsilon > 0$ such that each initial value in $\mathcal{E}(u_r + \varepsilon)$ determines a fixed point of T_r, then every initial value in $\mathcal{E}(u_\ell)$ determines a fixed point of T_r.

We now turn to showing the existence of fixed points of T_r with initial values close to u_r. We do this in stages.

To begin, we strengthen Condition E by imposing the stronger shock condition

$$f'(u_r) < s < f'(u_\ell) \ .$$

This relation with $(\leq)$ replacing $(<)$ is a result of Condition E. We will later weaken the strict inequality to $(\leq)$ by taking a limit.

We now establish

Lemma 2.13. If $s > f'(u_r)$, then it is possible to choose $\varepsilon > 0$ so that each initial value ξ_x for which

$$u_r - \varepsilon \le \xi_x \le u_r + \varepsilon$$

determines a fixed point of T_r.

Proof: Define the set $\mathcal{F}_{\alpha,\beta}$ to be the set of lattice functions with domain I^+, which satisfy

$$|v_x - u_r| \le \alpha e^{-\beta x}$$

where $\alpha, \beta > 0$. Let $v_x \in \mathcal{F}_{\alpha,\beta}$, then it follows from the mean value theorem that

$$(T_r v)_{x+\eta} e^{\beta(x+\eta)} = \sum_{j=-k}^{k} c_j(v_x, u_r) e^{-\beta j + \beta\eta} \, v_{x+j} e^{\beta(x+j)} \, .$$

The c_j's are positive as a result of the assumption of the monotonicity of $G(\cdot,\cdot,\ldots,\cdot)$. Applying the triangle inequality under the assumption that $v_x \in \mathcal{F}_{\alpha,\beta}$ we obtain

$$\left| (T_r v)_{x+\eta} e^{\beta(x+\eta)} \right| \le \alpha h(\alpha,\beta)$$

where

$$h(\alpha,\beta) = \sup_{\{v_x : |v_x - u_r| < \alpha\}} \left(\sum_{j=-k}^{k} c_j(v_x; u_r) e^{-\beta j + \beta\eta} \right) .$$

An easy calculation yields that

$$h(0,\beta) = (1 - e^{\beta}) e^{\beta\eta} \left\{ 1 - \theta \sum_{j=-k+1}^{k} \frac{\partial g}{\partial u_j} (u_r, u_r, \ldots, u_r) e^{-\beta j} \right\} \, .$$

A Taylor series expansion in β gives

$$h(0,\beta) = 1 - \theta(s - f'(u_r))\beta + O(\beta^2) \, .$$

We have used $f'(u) = \sum_j \frac{\partial g}{\partial u_j}(u,\ldots,u)$, which follows from $f(u) = g(u,u,\ldots,u)$. Hence, as $\theta(s - f'(u_r) > 0$ we may choose a $\beta > 0$ so that $h(0,\beta) < 1$. Fix this choice of β and choose $\alpha > 0$, so that $h(\alpha,\beta) < 1$. For this choice of α and β, we have shown that T_r maps $\mathcal{F}_{\alpha,\beta}$ into itself. $\mathcal{F}_{\alpha,\beta}$ is convex and compact in the topology of uniform convergence. T_r has a fixed point by the Schauder fixed point theorem. The lemma is proved.

Lemma 2.3 is established under the additional constraint $f'(u_r) < s$. Lemma 2.4 is true by symmetry. The analogous additional constraint is $s < f'(u_\ell)$. Theorem 1 is established for every choice of $f(\cdot)$ so that

$$f'(u_r) < s < f'(u_\ell) .$$

We now relax the inequalities. We consider the traveling waves to be fixed points of an operator T which depends on g. Note $g(u,u,\ldots,u) = f(u)$. We abuse the notation and write $T = T(f)$.

Suppose that f satisfies Condition E and the inequalities

$$f'(u_r) \leq s < f'(u_\ell) .$$

Choose ϕ_ε positive and smooth on $[u_r,u_\ell]$ so that ϕ_ε vanishes on the complement of $(u_r,u_r+\varepsilon)$. Set $f_\varepsilon = f + \phi_\varepsilon$ and

$$g_\varepsilon(u_1,u_2,\ldots,u_{2k}) = g(u_1,u_2,\ldots,u_{2k}) + \sum_{j=1}^{2k} \phi_\varepsilon(u_j)/2k .$$

Furthermore, choose ϕ_ε so that $G_\varepsilon(\cdot,\cdot,\ldots,\cdot)$ is monotone,

$$f_\varepsilon'(u_r) < s < f'(u_\ell) ,$$

and so that $f_\varepsilon \to f$ on $[u_r, u_\ell]$. Fix $u_o \in (u_r, u_\ell)$. There exist functions u_x^ε so that $u_o^\varepsilon = u_o$ and

$$T(f_\varepsilon)u_x^\varepsilon = u_x^\varepsilon .$$

Moreover, there is a fixed point w_x of $T_\ell(f_\varepsilon)$ with initial value identically equal to u_o which is independent of ε as $T_\ell(f_\varepsilon) = T_\ell(f)$ for lattice functions whose range lies above $u_r+\varepsilon$. Hence

$$w_x \leq u_x^\varepsilon \leq u_\ell \qquad (2.6)$$

for $x \leq 0$. As $u_x \in [u_r, u_\ell]$ pass to a subsequence, if necessary, so that for each value of x, $u_x^\varepsilon \to u_x$. Then $T(f)u_x = u_x$. As u_x^ε is monotone, u_x is also. The limits $u_{-\infty}$, $u_{+\infty}$ exist; $u_{-\infty}$ equals u_ℓ by (2.6). Moreover, we obtain

$$s(u_{+\infty} - u_\ell) = f(u_{+\infty}) - f(u_\ell)$$

as in the proof of Lemma 2.12. Hence $u_{+\infty} = u_r$.

We now have

Lemma 2.14. Theorem 1 is established for every $f(\cdot)$ which satisfies Condition E across $[u_r, u_\ell]$ and

$$f'(u_r) \leq s < f'(u_\ell) .$$

We now relax the last inequality.

Lemma 2.15. If $s \geq f'(u_r)$ and there exists a value u_ℓ so that $f(\cdot)$ satisfies Condition E across $[u_r, u_\ell]$ then it is possible to choose $\varepsilon > 0$, so that every initial value in $\mathcal{E}(u_r + \varepsilon)$ determines a fixed point of T_r.

Proof: It is possible to change $f(\cdot)$ to f_δ near u_ℓ so that f_δ satisfies Condition E across $[u_r, u_\ell]$ and so that

$$f'_\delta(u_\ell) > s \ .$$

Then there exists a fixed point of $T(f_\delta)$ which takes on the value $(u_r + u_\ell)/2$ at $x = 0$. This traveling wave is a fixed point of $T_r(f)$ on I^+. Lemma 2.8 completes the proof of this lemma.

Moreover, Lemma 2.3 is proved in complete generality. Lemma 2.4 is true by symmetry. Theorem 1 is established for η rational. In the next section we show these traveling waves are stable as a sequence η_i of rational numbers converges to an irrational number.

3. η Irrational

A good difference scheme for calculating approximations to discontinuous solutions of a partial differential equation produces approximations which change from values near u_ℓ to values near u_r in a few mesh widths. Experience obtained from numerical experiments indicates the behavior of the transition between u_ℓ and u_r for traveling waves is typical of the behavior of approximations to shocks generated by the finite difference scheme for an arbitrary initial condition. The results of Theorem 1 imply that the traveling wave $u_x(u_o)$ is a function of x and u_o, the value of u_x at $x = 0$. We produce this function explicitly for the special equation (1.9) with $u_r = -1$ and $u_\ell = +1$. The speed s of the traveling wave is zero, and the equation for the traveling wave is

$$u_x = \frac{u_{x+1} + u_{x-1}}{2} - \frac{\theta}{2}\left(u_{x+1}^2 - u_{x-1}^2\right) . \tag{3.1}$$

Equation (2.5) specialized to this equation is

$$\frac{u_x}{2} = \frac{u_{x+1}}{2} - \frac{\theta}{2}\left(u_{x+1}^2 + u_x^2\right) + \theta . \tag{3.2}$$

This is a quadratic relation which can be solved explicitly for u_{x+1} as a function of u_x. We set $u_o = 0$ and solve for u_1

$$u_1(\theta) = + \frac{1 - \sqrt{1 + 8\theta^2}}{2\theta} . \tag{3.3}$$

We choose the root so that $u_1(\theta) < 0$. The Courant-Friedrichs-Lewy condition for this equation is $\theta \leq \frac{1}{2}$. As θ varies from 0 to $\frac{1}{2}$, $u_1(\theta)$ varies from 0 to

$$-\sqrt{3} + 1$$

and is monotone in θ. This substantiates the empirical observation that the profile of a traveling wave becomes steeper as θ is increased to the limit imposed by the Courant-Friedrichs-Lewy condition. We do not establish this result in general. We establish a weaker result. We show that the traveling waves we have constructed converge to a monotone profile as η_i runs through rational values to an irrational limit. The limiting shape is defined on the entire real line and satisfies

$$u_{x+\eta} = G(u_{x+k}, u_{x+k-1}, \ldots, u_{x-k}) \tag{3.4}$$

and has limits $u_{+\infty} = u_r$, and $u_{-\infty} = u_\ell$.

When η is rational the minimal domain over which (3.4) makes sense is the linear span over the integers of 1 and η. The distance between adjacent points in the lattice is a function of η, which we call $\Delta(\eta)$. As η converges to an irrational number through rational values, $\Delta(\eta) \to 0$. Let $u_x(u_o;\eta_i)$ be the traveling wave which satisfies (3.4) with $\eta = \eta_i$ and takes on the value u_o at $x = 0$. We will construct a traveling wave as the limit of $u_x^i = u_x(u_o;\eta_i)$ as $\eta_i \to \eta$. Fix η irrational and suppose η_i, rational, converges to η. Choose a sequence s_i such that $\eta_i/s_i = \theta$ is constant and such that

$$s_i(u_\ell^i - u_r) = f(u_\ell^i) - f(u_r)$$

and f satisfies Condition E across $[u_r, u_\ell^i]$. If necessary, f can be extended on $[u_\ell, u_\ell+\varepsilon]$. We define v_x^i on the entire real line as piecewise constant. It is a consequence of the partial ordering and the maximum principle that a fixed point $\overline{u}_x$ of T_r with $\eta = \eta^+ < \eta^i$ for all i and which satisfies $\overline{u}_{2k} = u_o$ satisfies

$$u_r \le u_x^i \le \overline{u}_x \ . \qquad (3.5)$$

Extract a subsequence, if necessary, so that $u_x^i \to u_x$ for all rational x. The limit u_x is a monotone function and we may assume, without loss that u_x is continuous from the left. Moreover, an integral form of (3.4) and the bounded convergence theorem implies u_x satisfies (3.4) for all x. The inequality (3.5) implies that $u_{+\infty} = u_r$. As $u_x = u_o$ at $x = 0$, it follows as in Lemma 2.12 that $u_{-\infty} = u_\ell$.

We have established the existence of a traveling wave when η is irrational which takes on the value u_o at $x = 0$. We state this as

Theorem 2. (η irrational) Under the same condition as Theorem 1, except that η is irrational, there exists a traveling wave v_x which is monotone and continuous. Any other continuous traveling wave is a translate of v_x.

Proof: To obtain the uniqueness, systematically substitute integral signs for the summation signs in the proof of Lemma 2.1. We proceed to show that the traveling wave constructed above is a continuous function of x.

Suppose v_x and w_x are solutions of (3.1) which are continuous from the left. Then

$$h(t) = \int_{-\infty}^{\infty} (v_{x+t} - w_x)dx$$

is a continuous function of t which converges to $+\infty$ and $-\infty$ as t goes to minus and plus infinity respectively. Hence, there exists a value of t so that

$$v_{x+t} = w_x .$$

As the value u_o was chosen arbitrarily it follows that the function u_x constructed above assumes every value and is therefore a continuous function of x. This completes the proof.

4. Conclusions and Extensions

A second order accurate scheme to approximate (1.1) is

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \{Q(u_{j+1}^n, u_j^n) - Q(u_j^n, u_{j-1}^n)\} . \tag{4.1}$$

where

$$Q(a,b) = \frac{f(a)+f(b)}{2} + \frac{\Delta t}{\Delta x} \cdot \frac{f'(a)+f'(b)}{2} \cdot (f(a)-f(b)) .$$

The coefficients of the linearized equation are not positive no matter how small $\Delta t/\Delta x$ is taken. In general, the linearized equation of a second order accurate scheme does not have coefficients of the same sign.

Numerical experiments indicate that (4.1) is a stable approximation to (1.1) and that traveling waves exist which propagate with the speed s determined by the Rankine-Hugoniot relations. The methods used above cannot be used to establish the existence of traveling waves for these schemes as a second order scheme is not monotone.

An explicit calculation in the case $s = 0$ for the scheme (4.1) results in the proof of the existence of multiple traveling waves which have limits $u_r = -1$ and $u_\ell = +1$. In this case, (2.5) reduces to a first order recurrence and as such defines an operator $\mathcal{R}$ determining v_{x+1} as a function of v_x. One can determine explicitly the set of points in the real line which when iterated under $\mathcal{R}$ converge to -1. The essentially different solutions occur from considering different branches of the function $\mathcal{R}$.

The reader should note that in general (2.5) defines v_{x+k} as a function of $v_{x+k-1}, \ldots, v_{x-k+1}$. By the usual construction, the relation can be considered to determine a function $\mathcal{R}$ from R^{2k-1} into itself. The point, $\hat{u}_r$, in R^{2k-1}, all of whose components equal u_r is a fixed point of $\mathcal{R}$. The set of points which when iterated under $\mathcal{R}$ converge to $\hat{u}_r$ is called the stable manifold of $\mathcal{R}$ at u_r, [7]. Lemma 2.3 is an explicit parametrization of the stable manifold of $\mathcal{R}$

at $\hat{u}_r$ for the class of monotone difference operators, and Lemma 2.4 is a parametrization of the stable manifold of $\mathcal{R}^{-1}$ at $\hat{u}_\ell$. Theorem 1 states that these two manifolds intersect. With this approach the existence of traveling waves can be established when u, f, and g take on values in R and u_r and u_ℓ are close. The details will appear in a subsequent paper.

References

[1] Burgers, J. M., "A Mathematical Model Illustrating the Theory of Turbulence," Advances in Applied Mechanics, Vol. 1, 1948, pp. 171-179.

[2] Conley, Charles, C., and Joel A. Smoller, "Viscosity Matrices for Two Dimensional Nonlinear Hyperbolic Systems," to appear.

[3] Courant, R., K. O. Friedrichs, and H. Lewy, "Uber die Partiellen Differenzialgleichungen der mathematischen Physik," Math. Ann., 100 (1928), pp. 32-74.

[4] Foy, Linus, R., "Steady State Solutions of Hyperbolic Systems of Conservation Laws," Comm. Pure Appl. Math., XVII, 1964, pp. 177-188.

[5] Glimm, J. and P. D. Lax, "Decay of Solutions of Nonlinear Hyperbolic Conservation Laws," Memoir of American Mathematical Society, No. 101, Providence, R. I., 1970.

[6] Glimm, J., "Solutions in the Large for Nonlinear Hyperbolic Systems of Equations," Comm. Pure Appl. Math., XVIII, 1965, pp. 697-715.

[7] Hartman, P., "A Lemma in the Structural Stability of Differential Equations," Proc. Amer. Math. Soc., 11, 1960, pp. 610-620.

[8] Hopf, E., "The Partial Differential Equation $u_t + uu_x = \varepsilon u_{xx}$," Comm. Pure Appl. Math., III, 1950, pp. 201-230.

[9] Lax, P. D., "The Initial Value Problem for Nonlinear Hyperbolic Equations in Two Independent Variables," Princeton University Annals of Math. Studies, No. 33.

[10] Lax, P. D., "Weak Solutions of Nonlinear Hyperbolic Equations and Their Numerical Computation," Comm. Pure Appl. Math., VII, 1954, pp. 159-193.

[11] Lax, P. D., "Hyperbolic Systems of Conservation Laws II," Comm. Pure Appl. Math., X, 1957, 537-566.

[12] Lax, P. D., "Invariant Functionals of Nonlinear Equations of Evolution," Proc. of International Conf. on Functional Analysis and Related Topics, Tokyo, April, 1969, pp. 240-251.

[13] Lax, P. D., "Integrals of Nonlinear Equations of Evolution and Solitary Waves," Comm. Pure Appl. Math., XXi, 1968, pp. 467-490.

[14] Lax, P. D., "On the Stability of Difference Approximations to Solutions of Hyperbolic Equations with Variable Coefficient," Comm. Pure Appl. Math., XIV, pp. 497-520.

[15] Lax, P. D. and B. Wendroff, "Systems of Conservation Laws," Comm. Pure Appl. Math., XIII, 217-237.

[16] Lax, P. D., "Nonlinear Hyperbolic Systems of Conservation Laws," Nonlinear Problems, Langer, ed., 1960, pp. 3-12.

[17] Nishida, T., "Global Solution for an Initial Boundary Value Problem of a Quasilinear Hyperbolic System," Proc. Japan Acad., 44, (1968), pp. 642-646.

[18] Richtmyer, R. D., and K. W. Morton, Difference Methods for Initial Value Problems, Interscience, New York, 1967.

[19] Schwartz, J. T., Nonlinear Functional Analysis, Courant Inst. Math. Sciences, New York Univ., 1965, p. 21.

[20] Oleinik, O. A., "Discontinuous Solutions of Nonlinear Differential Equations," Uspehi Mat. Nauk., 1957, pp. 3-73, English Translation in Amer. Math. Soc. Trans., Ser. 2, No. 26, pp. 95-172.

[21] Vvedenskaya, N. D., "The Difference Method Solution of Cauchy's Problem for a Nonlinear Equation with Discontinuous Initial Values," Doklady Akad. Nauk SSSR, 3, 1956, pp. 517-521.

[22] Quinn, B. Keyfitz, "Time Decreasing Functionals of Nonlinear Conservations Laws," Comm. Pure Appl. Math., XXIV, 1971, pp. 125-132.

[23] Oleinik, O. A., "Uniqueness and Stability of the Generalized Solution of the Cauchy Problem for a Quasi-linear Equation," Uspehi Mat. Nauk., 1959, No. 2 (86), pp. 165-170, English Translation in Amer. Math. Soc. Trans., Ser. 2, No. 33, pp. 285-290.

[24] Lax, P. D., "The Formation and Decay of Shock Wave," Amer. Math. Monthly, Vol. 69 (3), March 1972, pp. 227-241.

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